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On the Leray–Schauder Formula and Bifurcation

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1. INTRODUCTION

In many applications of the topological degree theory it is essential to be able to calculate the exact value of the degree function. However, this is generally a hard problem to solve. In this paper we shall prove a generalization of the famous Leray–Schauder formula (see [Kr]) by which we obtain the value of degree for some linear operators. We shall apply these results to bifurcation problems.

A standard method is to calculate the value of the degree as an index of the gradient operator at an isolated critical point. For results in this direction we refer to [Am] in the finite dimensional case and in a more general setting to [Sk] and the references therein. Some special linear operators are treated in [BM2, 3, 6] and [Be3]. These results can be used in the study of nonlinear operator equations in various settings to prove the existence of nontrivial and multiple solutions (see [Sk], [DKN], [BM2, 3, 4], for instance).

Essential tool in this paper is the topological degree constructed in [BM1]. Considering systems of equations a rich variety of invertible linear maps is achieved for which we obtain a formula for the value of the degree (Theorem 4.1) (cf. [BM6, 7], [BMT]). In Section 5 we consider a system of wave- and beam operators with linear coupling and damping having the form

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\beta_1\frac{\partial u}{\partial t}-a_{11}u-a_{12}v=h_1(t,x) && \text{in } \Omega,\\
&\frac{\partial^2 v}{\partial t^2}+\frac{\partial^4 v}{\partial x^4}+\beta_2\frac{\partial v}{\partial t}-a_{21}u-a_{22}v=h_2(t,x) && \text{in } \Omega,\\
&u(t,0)=u(t,\pi)=0, && t\in\,]0,2\pi[,\\
&v(t,0)=v(t,\pi)=v_{xx}(t,0)=v_{xx}(t,\pi)=0, && t\in\,]0,2\pi[,\\
&u(\cdot,x),\ v(\cdot,x)\text{ are }2\pi\text{-periodic in }t,
\end{aligned}
$$

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where $\Omega =]0, 2\pi[\times]0, \pi[$ and $\beta_1 \geqslant 0, \beta_2 \geqslant 0$. Above the coupling matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The corresponding degree depends on the parameters $\beta_1, \beta_2, a_{11}, a_{12}, a_{21}$, and a_{22} and it is given in Theorem 5.1.

In Section 6 we illustrate the use of the formula of Theorem 4.1 by some new bifurcation results for systems of equations. We shall give a variant of the classical result, where we replace the eigenvalues of the differential operator by a more general set of parameters (cf. [La1], [LM1], [LM2]). The results given in this note are of a local nature. Both local and global bifurcation phenomena have been widely studied by several authors. In 1964, Krasnoselski [Kr] gave sufficient conditions for the local bifurcation for a class of operators of Leray–Schauder type, i.e., compact perturbations of identity. In 1971, Rabinowitz [Ra] deduced local and global bifurcation results using the topological Leray–Schauder degree and change of degree argument. Since then many variations and extensions of the classical results have been given. We refer here to the works of Stuart and Toland [ST], Laloux [La1], [La2], Laloux and Mawhin [LM1], [LM2], [La], Webb and Welsh [WW], Toland [To], Welsh [We1], [We2], [We3], Pascali [Pa], Drabek and Huang [DH], and the references therein.

2. PREREQUISITES

Let H be a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We recall some basic definitions.

A mapping $F: H \to H$ is

— *bounded* if it takes any bounded set into a bounded set,

— *demicontinuous* if $u_j \to u$ (norm convergence) implies $F(u_j) \rightharpoonup F(u)$ (weak convergence),

— *monotone* if $\langle F(u) - F(v), u - v \rangle \geqslant 0$ for all $u, v \in H$,

— *strongly monotone* if there exists $\alpha > 0$ such that $\langle F(u) - F(v), u - v \rangle \geqslant \alpha \|u - v\|^2$ for all $u, v \in H$,

— *of class* (S_+) if for any sequence with $u_j \rightharpoonup u$, $\limsup \langle F(u_j), u_j - u \rangle \leqslant 0$, it follows that $u_j \to u$,

— *quasimonotone* if for any sequence $u_j \rightharpoonup u$, $\limsup \langle F(u_j), u_j - u \rangle \geqslant 0$.

Let $L: D(L) \subset H \to H$ be a densely defined closed, normal linear operator with $\operatorname{Im} L = (\operatorname{Ker} L)^\perp$. The inverse K of the restriction of L to $\operatorname{Im} L \cap D(L)$ is a bounded linear operator on $\operatorname{Im} L$. We shall assume that K is compact.

Note that Ker L may be infinite dimensional. We are interested in the case where L is not self-adjoint and therefore we include the complex spectrum of L in our consideration. We recall that the complexification $H_C = H + iH$ of H has the usual linear structure and inner product induced by H, that is,

$$\langle u+iv, x+iy\rangle_{\mathbb{C}} = \langle u, x\rangle + \langle v, y\rangle + i(\langle v, x\rangle - \langle u, y\rangle)$$

for all $u+iv, x+iy \in H_{\mathbb{C}}$. For each $w = u+iv \in H_{\mathbb{C}}$ it is natural to denote $\bar{w} = u - iv$. We define the complex linear operator $L_{\mathbb{C}}: D(L_{\mathbb{C}}) \subset H_{\mathbb{C}} \to H_{\mathbb{C}}$ by setting $D(L_{\mathbb{C}}) = D(L) + iD(L)$ and $L_{\mathbb{C}}(u+iv) = Lu + iLv$ for all $u+iv \in D(L_{\mathbb{C}})$. It is clear that $\operatorname{Im} L_{\mathbb{C}} = (\operatorname{Ker} L_{\mathbb{C}})^{\perp}$, $L_{\mathbb{C}}$ is normal and its partial inverse $K_{\mathbb{C}}$ is compact. Consequently, $L_{\mathbb{C}}$ has a pure point spectrum $\sigma_{\mathbb{C}}(L) = \{\mu_j\}_{j \in \mathbb{Z}}$ with the corresponding orthonormal basis $\{\phi_j\}_{j\in\mathbb{Z}}$ of $H_{\mathbb{C}}$ such that

$$L_{\mathbb{C}}\phi_j = \mu_j \phi_j \qquad \text{for all} \quad j \in \mathbb{Z}.$$

Note that for any complex eigenvalue $\mu = \alpha + i\beta$ its complex conjugate $\bar{\mu} = \alpha - i\beta$ is also an eigenvalue.

For each $z = u+iv \in D(L_{\mathbb{C}})$ we have the spectral representation

$$L_{\mathbb{C}} z = \sum_{j\in\mathbb{Z}} \mu_j \langle z, \phi_j\rangle_{\mathbb{C}}\, \phi_j.$$

Denote by P and $Q = I - P$ the orthogonal projections to Ker L and $\operatorname{Im} L = (\operatorname{Ker} L)^{\perp}$, respectively. For any map $N: H \to H$ the equation

$$Lu - N(u) = 0, \qquad u \in D(L)$$

can be written equivalently as

$$Q(u - KQN(u)) + PN(u) = 0, \qquad u \in H,$$

where $K = (L|_{\operatorname{Im} L \cap D(L)})^{-1}: \operatorname{Im} L \to \operatorname{Im} L$ is assumed to be compact. Above we have used the fact that $KQ - P$ is the right inverse of $L - P$. If N is bounded, demicontinuous and of class (S_+), then there exists a topological degree for mappings of the form $F = Q(I + C) + PN$, where C is compact; see [BM1]. In fact, it is sufficient that N is bounded and demicontinuous and there exists an auxiliary map $\tilde{N}: H \to H$ such that $PN = P\tilde{N}$, $\tilde{N}$ is bounded demicontinuous and of class (S_+). This observation is quite obvious, since only the P-component of N appears in F. However, it has some interesting implications (see Lemma 3.1). The degree theory given in [BM1] is a unique extension of the classical Leray–Schauder degree. It is single valued and has the usual properties of degree, such as additivity and invariance under homotopies. Let the corresponding degree function be d_H.

In order to simplify our notations we define a further degree function $\deg_H$ by setting

$$\deg_H(L-N, G, 0) \equiv d_H(Q(I-KQN)+PN, G, 0)$$

for any open set $G \subset H$ such that $0 \notin (L-N)(\partial G \cap D(L))$. In the remainder of the paper we shall use the term admissible map to refer to any such map for which the degree is well defined. We use term admissible homotopy similarly. We shall indicate by subscript the space where the degree is calculated. If $V \subset H$ is a finite dimensional subspace, then d_V is reduced to the classical Brouwer degree. In some cases d_V will coincide with the classical Leray–Schauder degree. In what follows we have $G = B$, an open unit ball which will be denoted by the same symbol in any space involved.

3. ON SYSTEMS

Let H be a real separable Hilbert space and denote $\mathscr{H} = H^n$ with $n \geqslant 2$. We assume that $L_k : D(L_k) \subset H \to H$ is a densely defined closed, normal linear operator with $\operatorname{Im} L_k = (\operatorname{Ker} L_k)^\perp$ for each $k = 1, 2, ..., n$. The inverse K_k of the restriction of each L_k to $\operatorname{Im} L_k \cap D(L_k)$ is a bounded linear operator on $\operatorname{Im} L_k$. We shall further assume that the inverse K_k of each L_k is compact. In order to include systems of equations into consideration we define the diagonal operator $\mathscr{L} : D(\mathscr{L}) \subset \mathscr{H} \to \mathscr{H}$ by setting

$$\mathscr{L}u = (L_1 u_1, L_2 u_2, ..., L_n u_n)^T, \qquad u = (u_1, u_2, ..., u_n)^T \in D(\mathscr{L}),$$

where $D(\mathscr{L}) = D(L_1) \times D(L_2) \times \cdots \times D(L_n)$. Now the complexification $\mathscr{H}_\mathbb{C} = \mathscr{H} + i\mathscr{H} = (H_\mathbb{C})^n$ and $\mathscr{L}$ and $\mathscr{L}_\mathbb{C}$ inherit the properties of the component operators. We shall use the notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm in any real Hilbert space and the subscript $\mathbb{C}$ whenever the space is complex. For simplicity we shall frequently use the same symbol for an operator and its complexification. Hence in both $\mathscr{H}$ and $\mathscr{H}_\mathbb{C}$ we have

$$\operatorname{Im} \mathscr{L} = \operatorname{Im} L_1 \times \operatorname{Im} L_2 \times \cdots \times \operatorname{Im} L_n = (\operatorname{Ker} \mathscr{L})^\perp,$$

where

$$\operatorname{Ker} \mathscr{L} = \operatorname{Ker} L_1 \times \operatorname{Ker} L_2 \times \cdots \times \operatorname{Ker} L_n.$$

The inverse $\mathscr{K} = \mathscr{L}^{-1} : \operatorname{Im} \mathscr{L} \to \operatorname{Im} \mathscr{L}$ is compact with $\mathscr{K}u = (K_1 u_1, ..., K_n u_n)^T$ for all $u = (u_1, u_2, ..., u_n)^T \in \operatorname{Im} \mathscr{L}$. We denote by $\mathscr{P}$ and $\mathscr{Q}$ the orthogonal projections onto $\operatorname{Ker} \mathscr{L}$ and $\operatorname{Im} \mathscr{L}$, respectively.

Let $\mathcal{N}: \mathcal{H} \to \mathcal{H}$ be a (possibly nonlinear) bounded demicontinuous map. As described in Section 2 a topological degree is available for all admissible maps of the form $\mathcal{L} - \mathcal{N}$. In this paper we are mainly interested in linear maps of the following type. Let $A = (a_{lk})$ be a real $n \times n$-matrix and $\mathcal{A}: \mathcal{H} \to \mathcal{H}$ the *constant multiplication operator* induced by A, that is, for any $u = (u_1, u_2, ..., u_n)^T \in \mathcal{H}$

$$\mathcal{A} u = w = (w_1, w_2, ..., w_n)^T,$$

with $w_l = \sum_{k=1}^n a_{lk} u_k$, $l = 1, 2, ..., n$. In the following A also denotes the linear operator in $\mathbb{R}^n$ and $\mathbb{C}^n$ corresponding to the matrix (a_{lk}). Clearly $\sigma(\mathcal{A}) = \sigma(A)$, a real point spectrum which may be empty if n is even. Similarly $\sigma_{\mathbb{C}}(\mathcal{A}) = \sigma_{\mathbb{C}}(A)$ for complex spectra. If the matrix A is strictly positive, it is not hard to prove that

$$\langle \mathcal{A} u, u \rangle \geqslant \alpha \, \|u^2\| \qquad \text{for all} \quad u \in \mathcal{H},$$

where $\alpha = \min\{(Ax, x)_{\mathbb{R}^n}; |x|_{\mathbb{R}^n} = 1\}$ is positive. Hence the operator $\mathcal{A}$ is of class (S_+). In order to tackle more specific situations we assume that $\dim \operatorname{Ker} L_k = \infty$ for $k = 1, 2, ..., p$ and $\dim \operatorname{Ker} L_k < \infty$ for $k = p+1, ..., n$, where $0 \leqslant p \leqslant n$. If $p = n$ we assume that $A > 0$ and if $p = 0$, no positivity is needed. For the general case, we formulate the condition:

(PC) The matrix $(a_{lk})_{l,k=1}^p$ is strictly positive.

We have the following useful result.

LEMMA 3.1. *Assume that* $1 \leqslant p \leqslant n$ *and the positivity condition* (*PC*) *holds. Then there exists a bounded linear operator* $\tilde{A}: \mathcal{H} \to \mathcal{H}$ *of class* (S_+) *such that* $\mathcal{P}\tilde{\mathcal{A}} = \mathcal{P}\mathcal{A}$.

Proof. If $p = n$ there is nothing to prove. Hence we can assume that $1 \leqslant p < n$. We shall consider first a special case:

(a) Assume that $\operatorname{Ker} L_k = \{0\}$ for $k = p+1, ..., n$. Then

$$\mathcal{P}u = (P_1 u_1, P_2 u_2, ..., P_p u_p, 0, 0, ..., 0)^T \qquad \text{for all} \quad u = (u_1, u_2, ..., u_n)^T \in \mathcal{H}.$$

We can write

$$A = \begin{pmatrix} A_p & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_p = (a_{lk})_{l,k=1}^{p} > 0$ and the sizes of the blocks A_{12}, A_{21}, and A_{22} are $p \times (n-p)$, $(n-p) \times p$ and $(n-p) \times (n-p)$, respectively. We take

$$\tilde{A} = \begin{pmatrix} A_p & A_{12} \\ -A_{12}^T & \alpha_p I \end{pmatrix},$$

where $\alpha_p = \min\{(A_p x, x)_{\mathbb{R}^p};\ |x|_{\mathbb{R}^p} = 1\}$ is positive. Denote the corresponding constant multiplication operator by $\tilde{\mathscr{A}}$. It is easy to see that $\mathscr{P}\tilde{\mathscr{A}} = \mathscr{P}\mathscr{A}$ and $\langle \tilde{\mathscr{A}} u, u \rangle \geqslant \alpha_p \|u\|^2$ for all $u \in \mathscr{H}$. Hence the proof of this special case is complete.

(b) In the general situation we can use the matrix defined in case (a). Indeed, denote now

$$\hat{A} = \begin{pmatrix} A_p & A_{12} \\ -A_{12}^T & \alpha_p I \end{pmatrix}$$

and take $\tilde{\mathscr{A}} = \hat{\mathscr{A}} + \mathscr{P}(\mathscr{A} - \hat{\mathscr{A}})$. Clearly $\hat{\mathscr{A}}$ is of class (S_+) and $\mathscr{P}(\mathscr{A} - \hat{\mathscr{A}})$ is compact and hence quasimonotone implying that the operator $\tilde{\mathscr{A}}$ is of class (S_+) (see [BM1]), for instance). Since $\mathscr{P}\tilde{\mathscr{A}} = \mathscr{P}\mathscr{A}$ the proof is complete.

Remark. It is important to realize the meaning of $\tilde{\mathscr{A}}$; it is only needed to guarantee the existence of the topological degree. All concrete calculations will be done with $\mathscr{A}$, not with $\tilde{\mathscr{A}}$.

We shall impose a further common eigenbasis assumption:

(CE) The operators L_k, $k = 1, 2, ..., n$, have a common complex eigenbasis $\{\psi_j\}_{j \in \Lambda}$.

We denote the corresponding complex eigenvalues by $\{\mu_j^{(k)}\}_{j \in \Lambda}$. Hence $L_k \psi_j = \mu_j^{(k)} \psi_j$ for all $j \in \Lambda$, $k = 1, 2, ..., n$. Although assumption (CE) is very restrictive from the general point of view it can be verified in many applications. It trivially holds in the case $L_1 = L_2 = \cdots = L_n$. Without loss of generality we can assume that the index set $\Lambda \subset \mathbb{Z}$ is such that $\psi_{-j} = \bar{\psi}_j$ whenever $\psi_j \notin H$. We also denote $\Lambda_{\mathbb{R}} = \{ j \in \Lambda \mid \psi_j \in H\}$. For any $z = (z_1, z_2, ..., z_n)^T \in \mathscr{H}_{\mathbb{C}}$ we denote $\vec{z}_j = \sum_{k=1}^{n} \langle z_k, \psi_j \rangle_{\mathbb{C}}\, e_k$, where $\{e_k\}$ is the standard basis of $\mathbb{R}^n$. Hence we can write

$$z = \sum_{j \in \Lambda} \vec{z}_j \psi_j \qquad \text{and} \qquad \|z\|^2 = \sum_{j \in \Lambda} |\vec{z}_j|^2_{\mathbb{C}^n}.$$

It is easy to see that for any $z=u+iv\in D(\mathscr{L})+iD(\mathscr{L})$ we have a quasidiagonal representation

$$(\mathscr{L}-\mathscr{A})(z)=\sum_{j\in\Lambda}[(M_j-A)\,\vec{z}_j]\,\psi_j,$$

where $M_j=\operatorname{diag}(\mu_j^{(1)},\mu_j^{(2)},...,\mu_j^{(n)})$. Note that for all $j\in\Lambda_{\mathbb{R}}$ the matrix M_j is real. We can write

$$\mathscr{P}z=\sum_{j\in\Lambda}(R_j\vec{z}_j)\,\psi_j,$$

where

$$R_j\vec{z}_j=\sum_{k=1,\,\mu_j^{(k)}=0}^{n}\langle z_k,\psi_j\rangle_{\mathbb{C}}\,e_k.$$

Hence we obtain a representation

$$(\mathscr{K}\mathscr{Q}-\mathscr{P})\,z=\sum_{j\in\Lambda}[(M_j-R_j)^{-1}\,\vec{z}_j]\,\psi_j.$$

By the above formulas we get under the assumption (CE) (cf. [BM6], [BMT]):

Lemma 3.2. *The operator $\mathscr{L}-\mathscr{A}$ is injective if and only if* $\det(M_j-A)\neq 0$ *for all $j\in\Lambda$. Moreover, an injective $\mathscr{L}-\mathscr{A}$ is onto if and only if* $\sup_j\|(M_j-A)^{-1}\|<\infty$.

Note that in case $L_k=L$ for all $k=1,2,...,n$, the injectivity condition in Lemma 3.2 can be written as $\sigma_{\mathbb{C}}(L)\cap\sigma_{\mathbb{C}}(A)=\emptyset$. We get a simple formula for the norm of $(\mathscr{L}-\mathscr{A})^{-1}$:

Lemma 3.3. *If the operator $\mathscr{L}-\mathscr{A}$ is bijective, then*

$$\|(\mathscr{L}-\mathscr{A})^{-1}\|=\sup_j\|(M_j-A)^{-1}\|<\infty.$$

In general, the operator $\mathscr{L}-\mathscr{A}$ can be injective without being surjective. However, in the special case $L_1=L_2=\cdots=L_n$, injectivity implies bijectivity; see [BM6]. We shall give a counterexample for the wave-beam system considered in Section 5 in this paper. Indeed, if

$$A=\begin{pmatrix}0 & -1\\ 0,5 & 0\end{pmatrix},$$

then it is easy to see that the wave-beam operator $\mathscr{L}-\mathscr{A}$ in $\mathscr{H}=(L_2(\Omega))^2$ is injective but not surjective. By condition (PC) we get the following nontrivial result.

LEMMA 3.4. *Assume (PC), i.e., the matrix* $(a_{lk})_{l,k=1}^{p}$, *is strictly positive. Then the injectivity of the operator* $\mathscr{L}-\mathscr{A}$ *implies its surjectivity.*

Proof. Assume that $\mathscr{L}-\mathscr{A}$ is injective. By the quasidiagonal representation and the above lemmas it is sufficient to prove that $(\mathscr{L}-\mathscr{A})^{-1}$ is bounded. Indeed, assume the contrary. Then there exists a sequence $(u_k)\subset D(\mathscr{L})$, $\|u\|_k=1$, such that $w_k=: \mathscr{L}u_k-\mathscr{A}u_k\to 0$. Without loss of generality we can assume that $u_k\rightharpoonup u$ in $\mathscr{H}$. The compactness of $\mathscr{K}$ implies that $\mathscr{Q}u_k\to\mathscr{Q}u$. If $\dim\operatorname{Ker}\mathscr{L}<\infty$, that is, $p=0$, the proof is complete. Assume that $p>0$. By Lemma 3.1 there exists a linear bounded operator $\tilde{A}:\mathscr{H}\to\mathscr{H}$ of class (S_+) such that $\mathscr{P}\tilde{\mathscr{A}}=\mathscr{P}\mathscr{A}$. Hence

$$\begin{aligned}\limsup\langle\tilde{\mathscr{A}}u_k, u_k-u\rangle &= \limsup\langle\mathscr{P}\mathscr{A}u_k, u_k-u\rangle\\ &= \lim\langle -\mathscr{P}w_k, u_k-u\rangle=0\end{aligned}$$

implying $u_k\to u$. Since $\mathscr{L}$ is closed, we get $\mathscr{L}u-\mathscr{A}u=0$ with $\|u\|=1$, a contradiction completing the proof.

Define the *matrix resolvent* (see [BMT]) of the operator $\mathscr{L}$ as the set

$$\rho_M(\mathscr{L}) = \{A\in\mathbb{R}_{n\times n} \mid \operatorname{Im}(\mathscr{L}-\mathscr{A}) \text{ is dense in } \mathscr{H} \text{ and } \mathscr{L}-\mathscr{A} \text{ has a bounded inverse}\}.$$

The *matrix spectrum* of the operator $\mathscr{L}$ is then the set

$$\sigma_M(\mathscr{L})=\{A\in\mathbb{R}_{n\times n}\mid A\notin\rho_M(\mathscr{L})\}.$$

In our case we can write

$$\sigma_M(\mathscr{L}) = \{A\in\mathbb{R}_{n\times n}\mid \mathscr{L}-\mathscr{A} \text{ is not injective or } (\mathscr{L}-\mathscr{A})^{-1} \text{ is not bounded}\}.$$

Assuming (CE), that is, the existence of a common complex eigenbasis, we can write $\sigma_M(\mathscr{L})=(\bigcup_{j\in\Lambda}\sigma_j)\cup\sigma_\infty$, where

$$\sigma_j=\{A\in\mathbb{R}_{n\times n}\mid \det(M_j-A)=0\},\qquad j\in\Lambda$$

and

$$\sigma_\infty=\{A\in\mathbb{R}_{n\times n}\mid A\notin\textstyle\bigcup_{j\in\Lambda}\sigma_j, \sup_j \|(M_j-A)\|^{-1}=\infty\}.$$

Now $\bigcup_{j \in \Lambda} \sigma_j$ corresponds to the usual point spectrum and σ_∞ to the continuous spectrum. By Lemma 3.4, $A \notin \sigma_\infty$ whenever the positivity condition (PC) holds for A. We shall close this section by a reduction result frequently used in this paper. It is a consequence of a more general Theorem 6.1 in [Be1]. The notation $\overline{\mathrm{sp}}_{\mathbb{C}}$ stands for the closure of the span over complex numbers. Recall that the operator $\mathscr{L} - \mathscr{A}$ is *completely reduced* by a closed linear subspace $V \subset \mathscr{H}$ if

$$P_V(D(\mathscr{L})) \subset D(\mathscr{L}) \qquad \text{and} \qquad (\mathscr{L} - \mathscr{A})\, P_V u = P_V(\mathscr{L} - \mathscr{A})\, u$$

for all $u \in D(\mathscr{L})$, where P_V is the orthogonal projection from $\mathscr{H}$ onto V.

Lemma 3.5. *Assume* (CE) *and* (PC). *Let $\Lambda_0 \subset \Lambda$ be any index set which is symmetric in the sense that for any $j \in \Lambda_0$ with $\psi_j \notin H$ it follows that $-j \in \Lambda_0$ (recall that $\psi_{-j} = \overline{\psi}_j$). Set*

$$V = \overline{\mathrm{sp}}_{\mathbb{C}}\{\psi_j e_k, \mid k = 1, 2, ..., n,\ j \in \Lambda_0\} \cap \mathscr{H}.$$

Then $\mathscr{L} - \mathscr{A}$ is completely reduced by V and if $A \notin \sigma_M(\mathscr{L})$, then

$$\begin{aligned} &\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) \\ &\quad = \deg_V((\mathscr{L} - \mathscr{A})|_V, B, 0)\, \deg_{V^\perp}((\mathscr{L} - \mathscr{A})|_{V^\perp}, B, 0). \end{aligned}$$

4. GENERALIZED LERAY–SCHAUDER FORMULA

We shall now calculate the value of the degree $\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0)$, where $A \notin \sigma_M(\mathscr{L})$. The common eigenbasis condition (CE) is assumed to hold throughout this section. As in Section 3 we suppose that $\dim \operatorname{Ker} L_k = \infty$ for $k = 1, 2, ..., p$ and $\dim \operatorname{Ker} L_k < \infty$ for $k = p+1, p+2, ..., n$, where $0 \leqslant p \leqslant n$. We shall assume that the positivity condition (PC) is satisfied. Note that in case $p = 0$ there is no positivity condition for A and in case $p = n$ we have the condition $A > 0$. Recall that in order to simplify the calculations we assume that $\psi_{-j} = \overline{\psi}_j$ whenever $\psi_j \notin H$. We also denote $\Lambda_{\mathbb{R}} = \{ j \in \Lambda \mid \psi_j \in H\}$. Recalling the definition of the degree we shall consider the function

$$\begin{aligned} \mathscr{F}(u) &= [\mathscr{Q}(I - \mathscr{K}\mathscr{Q}\mathscr{A}) + \mathscr{P}\mathscr{A}](u) \\ &= \sum_{j \in \Lambda} (M_j - R_j)^{-1} (M_j - A)\, \vec{u}_j \psi_j, \qquad u \in \mathscr{H}. \end{aligned}$$

For the following lemmas we define two different types of finite dimensional subspaces of $\mathscr{H}$. First, if $\psi_j \notin H$, then $\psi_{-j} = \bar{\psi}_j$ and we set

$$W_j = \mathrm{sp}_{\mathbb{C}}\{\psi_j e_k, \psi_{-j} e_k \mid k = 1, 2, ..., n\} \cap \mathscr{H},$$

where $\mathrm{sp}_{\mathbb{C}}$ stands for the complex span of a given set of vectors. Similarly, the real span is denoted $\mathrm{sp}_{\mathbb{R}}$ or simply sp if there is no ambiguity. Note that $\mathrm{sp}_{\mathbb{C}}\{\psi_j e_k \mid k = 1, 2, ..., n\} \cap \mathscr{H} = \{0\}$ above. In case $\psi_j \in H$, we set

$$V_j = \mathrm{sp}_{\mathbb{R}}\{\psi_j e_k \mid k = 1, 2, ..., n\}.$$

It is easy to see that W_j and V_j are invariant subspaces under $\mathscr{F}$ and $\mathscr{F}$ is completely reduced by any W_j or V_j. We start with

LEMMA 4.1. *Assume that $\psi_j \notin H$, $\psi_{-j} = \bar{\psi}_j$ and $A \notin \sigma_M(\mathscr{L})$. Denote by $\mathscr{F}_j$ the restriction of $\mathscr{F}$ to the subspace W_j of $\mathscr{H}$. Then*

$$d_{W_j}(\mathscr{F}_j, B, 0) = +1.$$

Proof. *For $u \in W_j$ we have*

$$\mathscr{F}_j(u) = (M_j - R_j)^{-1} (M_j - A) \vec{u}_j \psi_j + (M_{-j} - R_{-j})^{-1} (M_{-j} - A) \vec{u}_{-j} \psi_{-j},$$

where $R_{-j} = R_j$ and $M_{-j} = \bar{M}_j$. Hence by the basic definition of the Brouwer degree (see [L1]) we get

$$\begin{aligned} & d_B(\mathscr{F}_j, B, 0) \\ &\quad = sgn\{\det[(M_j - R_j)^{-1} (M_j - A)] \det[(M_{-j} - R_j)^{-1} (M_{-j} - A)]\} \\ &\quad = sgn\, |\det(M_j - A)|^2\ |\det(M_j - R_j)|^2 = +1 \end{aligned}$$

completing the proof.

Our next result obviously follows from the basic definition of the Brouwer degree.

LEMMA 4.2. *Assume that $\psi_j \in H$ and $A \notin \sigma_M(\mathscr{L})$. Denote by $\mathscr{F}_j$ the restriction of $\mathscr{F}$ to the subspace V_j of $\mathscr{H}$. Then*

$$d_{V_j}(\mathscr{F}_j, B, 0) = sgn[\det(M_j - A) \det(M_j - R_j)].$$

Of course, it is essential to find those real subspaces V_j where the corresponding degree is -1. Indeed, we shall prove that there exists only a finite number, say χ, of subspaces with negative degree and $\deg_{\mathscr{H}}(\mathscr{L}-\mathscr{A}, B, 0)=(-1)^{\chi}$. The crucial result is the following, where the condition (PC) is actually needed.

LEMMA 4.3. *Assume that $A \notin \sigma_M(\mathscr{L})$ and A satisfies the positivity condition (PC). Then the number of j's such that*

$$sgn \det(M_j-A)\det(M_j-R_j)=-1, \qquad j\in \Lambda_{\mathbb{R}}$$

is finite.

Proof. Assume the contrary. Then we can find a sequence (V_{j_k}) of subspaces such that

$$d_{V_{jk}}(\mathscr{F}_{V_{jk}}, B, 0)=-1 \qquad \text{for all} \quad k=1, 2, \ldots.$$

Hence there exist sequences (u_k), $u_k \in V_{j_k}$, $\|u\|_k \equiv 1$, and (t_k), $0<t_k<1$, such that

$$(1-t_k)\,u_k+t_k\mathscr{F}(u_k)=0 \qquad \text{for all} \quad k\in \mathbb{Z}_+.$$

Without loss of generality we can assume that $t_k \to t\in[0,1]$ and $u_k \rightharpoonup u=0$. From the above equation we get

$$\mathscr{Q}u_k=t_k\mathscr{K}\mathscr{Q}\mathscr{A}(u_k)\to t\mathscr{K}\mathscr{Q}\mathscr{A}(u)=\mathscr{Q}u=0$$

and

$$(1-t_k)\,\mathscr{P}u_k+t_k\mathscr{P}\mathscr{A}(u_k)\equiv 0.$$

By Lemma 3.1 there exists a linear bounded operator $\tilde{A}: \mathscr{H}\to\mathscr{H}$ of class (S_+) such that $\mathscr{P}\tilde{\mathscr{A}}=\mathscr{P}\mathscr{A}$. Thus

$$\limsup \langle \tilde{\mathscr{A}}(u_k), u_k-u\rangle=\limsup\left\{-\frac{1-t_k}{t_k}\,\|\mathscr{P}u_k\|^2\right\}\leqslant 0$$

implying $u_k\to 0$. But this is a contradiction, which completes the proof.

We define now two complementary subspaces of $\mathscr{H}$ by setting

$$E_-=\operatorname{sp}\{\psi_j e_k \mid j\in\Lambda_{\mathbb{R}}, \det(M_j-A)\det(M_j-R_j)<0, k=1, 2, \ldots, n\}$$

and $E_+=E^{\perp}$. By the preceding lemmas, the space E_- is finite dimensional and E_+ contains all subspaces W_j and those V_j's where the corresponding degree is $+1$.

LEMMA 4.4. *Assume that $A \notin \sigma_M(\mathscr{L})$ and (PC) holds. Denote by $\mathscr{F}_+$ the restriction of $\mathscr{F}$ to the subspace E_+ defined above. Then*

$$d_{E_+}(\mathscr{F}_+, B, 0) = +1.$$

Proof. By the definition of the space E_+ we can find a family of subspaces $E_k, k \in \mathbb{Z}_+$, of E_+ such that $E_k \subset E_{k+1}$, $\bigcup_{k=1}^{\infty} E_k$ is dense in E_+, the map $\mathscr{F}_+$ is completely reduced by E_k, and $d_{E_k}(\mathscr{F}_{E_k}, B, 0) = +1$ for all $k \in \mathbb{Z}_+$. We shall argue by contradiction. Indeed, we assume that $d_{E_+}(\mathscr{F}_+, B, 0) \neq +1$. Then by Lemma 3.5

$$d_{E_+}(\mathscr{F}_+, B, 0) = d_{E_k}(\mathscr{F}_{E_k}, B, 0)\, d_{E_k^\perp}(\mathscr{F}_{E_k^\perp}, B, 0) = d_{E_k^\perp}(\mathscr{F}_{E_k^\perp}, B, 0) \neq +1.$$

Hence there exist sequences $(u_k), u_k \in E_k^\perp$, $\|u\|_k = 1$, and $(t_k), 0 < t_k < 1$, such that

$$(1 - t_k)\, u_k + t_k\, \mathscr{F}(u_k) = 0 \qquad \text{for all} \quad k \in \mathbb{Z}_+.$$

As in the proof of Lemma 4.3 we can assume that $t_k \to t \in [0, 1]$ and $u_k \rightharpoonup u = 0$. Proceeding as in the above proof we obtain a contradiction $u_k \to 0$, which completes the proof.

We eventually get our main result, a generalized Leray–Schauder formula.

THEOREM 4.1. *Assume that $A \notin \sigma_M(\mathscr{L})$ and conditions (PC) and (CE) hold. Then*

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = (-1)^{\chi},$$

where

$$\chi = \#\{ j \in \Lambda_{\mathbb{R}} \mid \det(M_j - R_j)\det(M_j - A) < 0\}.$$

Proof. By Lemmas 3.5 and 4.4 we have

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = d_{E_+}(\mathscr{F}_+, B, 0)\, d_{E_-}(\mathscr{F}_-, B, 0) = d_{E_-}(\mathscr{F}_-, B, 0),$$

where $\mathscr{F}_+$ and $\mathscr{F}_-$ stand for the restrictions of the map $\mathscr{F}$ to subspaces E_+ and E_-, respectively. Since E_- is a direct sum of finite number χ of subspaces V_j with corresponding degree -1 we get via Lemma 3.5 the desired result

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0 = d_{E_-}(\mathscr{F}_-, B, 0) = (-1)^{\chi}$$

completing the proof.

5. WAVE-BEAM SYSTEM

In order to illuminate the use of Theorem 4.1 we shall consider a system of a wave equation and a beam equation with linear coupling and damping having the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \beta_1 \frac{\partial u}{\partial t} - a_{11} u - a_{12} v = h_1(t, x) \qquad \text{in} \quad \Omega,$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} + \beta_2 \frac{\partial v}{\partial t} - a_{21} u - a_{22} v = h_2(t, x) \qquad \text{in} \quad \Omega,$$

$$u(t, 0) = u(t, \pi) = 0, \qquad t \in]0, 2\pi[,$$

$$v(t, 0) = v(t, \pi) = v_{xx}(t, 0) = v_{xx}(t, \pi) = 0, \qquad t \in]0, 2\pi[,$$

$$u(\cdot, x),\ v(\cdot, x) \text{ are } 2\pi\text{-periodic in } t,$$

where $\Omega =]0, 2\pi[\times]0, \pi[$ and $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$. Here, the coupling matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Denoting $H = L_2(\Omega)$ and $\phi_{jk}(t, x) = \frac{1}{\pi} \sin(jx) \exp(ikt)$, $(t, x) \in \Omega$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, the set $\{\phi_{jk}\}$ forms an orthonormal basis in $H_{\mathbb{C}}$. Observe the natural indexing used here. The wave operator $L_1 = \partial_{tt} - \partial_{xx} + \beta_1 \partial_t$ has in $H_{\mathbb{C}}$ the representation

$$L_1 u = \sum_{j,k} (\lambda_{jk}^{(1)} + i\beta_1 k) \langle u, \phi_{jk} \rangle_{\mathbb{C}}\, \phi_{jk}, \qquad u \in D(L_1)$$

with $\lambda_{jk}^{(1)} = j^2 - k^2$ and

$$D(L_1) = \left\{ u \in H_{\mathbb{C}} \,\middle|\, \sum_{j,k} |\lambda_{jk}^{(1)} + i\beta_1 k|^2\, |\langle u, \phi_{jk} \rangle_{\mathbb{C}}|^2 < \infty \right\}.$$

If $\beta_1 > 0$, then L_1 is normal and $\operatorname{Ker} L_1 = \{0\}$. If $\beta_1 = 0$ then L_1 is self-adjoint and $\operatorname{Ker} L_1$ is infinite dimensional. The spectrum of L_1 is $\sigma_{\mathbb{C}}(L_1) = \{ j^2 - k^2 + i\beta_1 k \mid j \in \mathbb{Z}_+, k \in \mathbb{Z} \}$. Note that in the real case $\beta_1 = 0$ the spectrum is unbounded from below and from above. For the beam operator $L_2 = \partial_{tt} + \partial_{xxxx} + \beta_2 \partial_t$ we have analogous representation

$$L_2 v = \sum_{j,k} (\lambda_{jk}^{(2)} + i\beta_2 k) \langle v, \phi_{jk} \rangle_{\mathbb{C}}\, \phi_{jk}, \qquad v \in D(L_2),$$

where $\lambda_{jk}^{(2)} = j^4 - k^2$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. If $\beta_2 > 0$, then the operator L_2 is normal, $\operatorname{Ker} L_2 = \{0\}$, and $\sigma_{\mathbb{C}}(L_2) = \{ j^4 - k^2 + i\beta_2 k \mid j \in \mathbb{Z}_+, k \in \mathbb{Z}\}$. Clearly L_2 is self-adjoint with infinite dimensional kernel and unbounded spectrum in the case $\beta_2 = 0$.

The diagonal operator $\mathscr{L} = \operatorname{diag}(L_1, L_2)$ is defined on $D(\mathscr{L}) = D(L_1) \times D(L_2) \subset \mathscr{H}_{\mathbb{C}} = H_{\mathbb{C}}^2$. Since L_1 and L_2 are normal both having compact (partial) inverses, also $\mathscr{L}$ is normal with compact inverse from $\operatorname{Im} \mathscr{L}$ into $\operatorname{Im} \mathscr{L}$. The above presented system corresponds to the linear equation

$$\mathscr{L} w - \mathscr{A} w = h, \qquad w = (u, v)^T \in D(\mathscr{L}) \subset \mathscr{H} = H^2,$$

where $h = (h_1, h_2)^T \in \mathscr{H}$.

Denote $M_{jk} = \operatorname{diag}(\lambda_{jk}^{(1)} + i\beta_1 k, \lambda_{jk}^{(2)} + i\beta_2 k)$. In view of Lemma 3.2 the linear operator $\mathscr{L} - \mathscr{A}$ is injective if and only if

$$\det(M_{jk} - A) = \det\begin{pmatrix} \lambda_{jk}^{(1)} + i\beta_1 k - a_{11} & -a_{12} \\ -a_{21} & \lambda_{jk}^{(2)} + i\beta_2 k - a_{22} \end{pmatrix} \neq 0$$

for all $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. Moreover, assuming the positivity condition (PC), injectivity implies surjectivity by Lemma 3.4.

Assume that $A \notin \sigma_M(\mathscr{L})$ and the positivity condition (PC) holds. Note that the matrix spectrum varies with the parameters β_1 and β_2. Since $\Lambda_{\mathbb{R}} = \{(j, k) \mid j \in \mathbb{Z}_+, k = 0\}$ in this case, we have the formula

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = (-1)^{\chi},$$

where

$$\chi = \#\{ j \in \mathbb{Z}_+ \mid \det(M_{j0} - R_{j0}) \det(M_{j0} - A) < 0\}.$$

Now $\lambda_{j0}^{(1)} = j^2$ and $\lambda_{j0}^{(2)} = j^4$ and hence $\det(M_{j0} - R_{j0}) = j^6 \geqslant 1$. Thus we get a simple formula

$$\chi = \#\{ j \in \mathbb{Z}_+ \mid j^6 - a_{11} j^4 - a_{22}\, j^2 + \det A < 0\}. \tag{5.1}$$

Thus we have obtained the following theorem.

THEOREM 5.1. *Assume that* $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$, $A \notin \sigma_M(\mathscr{L})$, *and* (1) $a_{11} > 0$ *if* $\beta_1 = 0$, $\beta_2 \neq 0$ (2) $a_{22} > 0$, *if* $\beta_1 \neq 0$, $\beta_2 = 0$, (3) $A > 0$ *if* $\beta_1 = \beta_2 = 0$ (4) *no positivity assumption, if* $\beta_1 \neq 0$, $\beta_2 \neq 0$. *Then the degree for the corresponding wave-beam operator is*

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = (-1)^{\chi},$$

where

$$\chi = \#\{j \in \mathbb{Z}_+ \mid j^6 - a_{11} j^4 - a_{22} j^2 + \det A < 0\}.$$

For instance, if

$$A = \begin{pmatrix} 35 & -42 \\ 125 & -150 \end{pmatrix},$$

then $j^6 - a_{11} j^4 - a_{22} j^2 + \det A = j^2(j^2-5)(j^2-30)$, which is negative if and only if $j=3$, $j=4$, or $j=5$. Thus $\chi=3$ and the corresponding degree is -1 whenever defined. Indeed, according to the above theorem there are four different cases:

(1) If $\beta_1 = 0$, $\beta_2 \neq 0$, then $A \notin \sigma_M(\mathscr{L})$ and

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = (-1)^{\chi} = (-1)^3 = -1.$$

(2) If $\beta_1 \neq 0$, $\beta_2 = 0$, then the operator $\mathscr{L} - \mathscr{A}$ is not admissible and the degree is not defined, since $a_{22} = -150 < 0$. However, the operator is injective.

(3) If $\beta_1 = \beta_2 = 0$, then the operator $\mathscr{L} - \mathscr{A}$ is not admissible and the degree is not defined. Moreover, $A \in \sigma_M(\mathscr{L})$ and hence the operator $\mathscr{L} - \mathscr{A}$ is not even injective.

(4) If $\beta_1 \neq 0$, $\beta_2 \neq 0$, then no positivity is needed, $A \notin \sigma_M(\mathscr{L})$ and

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}, B, 0) = (-1)^3 = -1.$$

6. BIFURCATION RESULTS

As an application of Theorem 4.1 we consider the existence of nontrivial solutions for the equation

$$\mathscr{L}u - \mathscr{A}_s u - \mathscr{N}(s, u) = 0, \qquad u \in D(\mathscr{L}),\ s \in \mathbb{R}, \tag{E}$$

where $\mathscr{N}\colon \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ is a nonlinear map with $\mathscr{N}(s, 0) \equiv 0$ and $\mathscr{A}_s$ is a constant multiplication operator generated by a parameter dependent matrix A_s. Other notations are adopted from Section 3 of this paper. We shall study how the solution set of (E) varies with the real parameter s. Let S be the set of nontrivial solutions of (E), i.e.,

$$S = \{(s, u) \in \mathbb{R} \times D(\mathscr{L}) \mid \mathscr{L}u - \mathscr{A}_s u - \mathscr{N}(s, u) = 0,\ u \neq 0\}.$$

We shall call $(s_0, 0)$ a bifurcation point for (E) if and only if there exists a sequence $((s_j, u_j))_{j=1}^{\infty} \subset S$ such that $s_j \to s_0$ and $\|u\|_j \to 0$. The definition above coincides with the standard concept in case $\mathscr{A}_s = sI$. It is well known that in the standard case the points of bifurcation are of the type $(s_0, 0)$, where s_0 is an eigenvalue of $\mathscr{L}$ having odd multiplicity; see [Ra], [Ma], [Be2], for instance.

We shall give a variant of the classical result (cf. [Be3], [La1], [LM1], [LM2]). We replace the eigenvalues of $\mathscr{L}$ by those values of the parameter s for which $A_s \in \sigma_M(\mathscr{L})$. The use of degree argument imposes further conditions on the matrix A_s as well as on the nonlinearity $\mathscr{N}$. Our results are of a local nature.

Let now $a_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j = 1, ..., n$, be continuous functions of a real parameter s and denote $A_s = (a_{ij}(s))$. Assume that the matrix A_s is strictly positive for all s in some bounded interval $[a, b]$. Then the corresponding operator $\mathscr{A}_s : \mathscr{H} \to \mathscr{H}$ is strongly monotone uniformly in s on $[a, b]$; i.e., there exists $\alpha > 0$ such that

$$\langle \mathscr{A}_s u, u \rangle \geqslant \alpha \|u\|^2 \qquad \text{for all} \quad u \in \mathscr{H}, s \in [a, b].$$

Clearly $\mathscr{A}_s$ defines a *continuous homotopy of class* (S_+) on $[a, b]$; i.e., if $u_k \rightharpoonup u$, $s_k \to s$, and $\limsup \langle \mathscr{A}_{s_k} u_k, u_k - u \rangle \leqslant 0$, then $u_k \to u$ and $\mathscr{A}_{s_k} u_k \to \mathscr{A}_s u$ (see [BM1]). Throughout this section we assume that condition (CE) holds. Essential to our study is the behavior of the continuous functions $f_j(s) =: \det(M_j - A_s)$, $j \in \Lambda$. If $A_s \notin \sigma_M(\mathscr{L})$ for all $a \leqslant s \leqslant b$, then the homotopy invariance property of the degree implies

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}_a, B, 0) = \deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}_b, B, 0).$$

The change of degree, which is crucial to the bifurcation, is only possible if the parameter s crosses a value s_0 such that $f_j(s_0) = 0$ and f_j changes sign at the point $s = s_0$ for some $j \in \Lambda_{\mathbb{R}}$. Denote

$$\Sigma = \bigcup_{j \in \Lambda} f_j^{-1}(0) = \{ s \in \mathbb{R} \mid A_s \in \sigma_M(\mathscr{L}) \}.$$

Clearly the set Σ is closed in $\mathbb{R}$. Note that in the standard case $A_s = sI$ we are dealing with the usual spectrum.

Let $\mathscr{N} : [a, b] \times \mathscr{H} \to \mathscr{H}$ be a bounded demicontinuous map, which is quasimonotone in the sense that $u_k \rightharpoonup u$, $s_k \to s$ imply $\limsup \langle \mathscr{N}(s_k, u_k), u_k - u \rangle \geqslant 0$. This holds, for instance, if $\mathscr{N}$ is compact. It is easy to see that the map $\mathscr{A}_s + \mathscr{N}(s, \cdot)$, $s \in [a, b]$, defines a homotopy of class (S_+). We assume in the following that

$$\|\mathscr{N}(s, u)\| = o(\|u\|) \qquad \text{as } u \to 0 \text{ uniformly in } s \text{ on } [a, b]. \qquad (6.1)$$

We have the following necessary condition for the existence of the bifurcation (for the proof, cf. Theorem 4.1 in [Be3]).

THEOREM 6.1. *Assume that* (6.1) *holds and* $(s_0, 0)$, $s_0 \in [a, b]$ *is a point of bifurcation for the equation*

$$\mathscr{L}u - \mathscr{A}_s u - \mathscr{N}(s, u) = 0, \qquad u \in D(\mathscr{L}),\ s \in \mathbb{R}. \tag{E}$$

Then $s_0 \in \Sigma$; *i.e.*, $A_{s_0} \in \sigma_M(\mathscr{L})$.

In case $A_s = sI$ the previous theorem states the well-known necessary condition for bifurcation that s_0 is an eigenvalue of $\mathscr{L}$. A typical sufficient condition is that the multiplicity of an eigenvalue of $\mathscr{L}$ is odd.

According to Theorem 4.1 we can write $\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}_s, B, 0) = (-1)^{\chi_s}$ for any $s \in [a, b]$ such that $A_s \notin \sigma_M(\mathscr{L})$.; i.e., $s \notin \Sigma$. In order to obtain a bifurcation result for equation (E), we shall first study the change of degree as s crosses an isolated point of Σ. Indeed, assume that $s_0 \in \Sigma \cap\,]a, b[$ and there exists $\delta > 0$ such that $]s_0 - \delta, s_0 + \delta[\, \cap \Sigma = \{s_0\}$. Hence $f_j(s) = \det(M_j - A_s) \neq 0$ for all $j \in \Lambda$ and $s \in\,]s_0 - \delta, s_0 + \delta[\setminus \{s_0\}$. The value of χ_s remains constant on $]s_0 - \delta, s_0[$ and $]s_0, s_0 + \delta[$, respectively. Denoting the corresponding constants by χ_- and χ_+ we thus have

$$\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}_s, B, 0) = \begin{cases} (-1)^{\chi_-}, & s_0 - \delta < s < s_0 \\ (-1)^{\chi_+}, & s_0 < s < s_0 + \delta. \end{cases}$$

Clearly there is an index jump, that is, a change of degree at s_0 if and only if $\chi_- + \chi_+$ is odd. Now we are ready to give our first bifurcation result.

THEOREM 6.2. *Assume that* (6.1) *holds and* $s_0 \in \Sigma \cap\,]a, b[$ *is an isolated point of* Σ. *If the number* $\chi_- + \chi_+$ *is odd, then* $(s_0, 0)$ *is a point of bifurcation for equation* (E).

The proof of the above result is standard; see for instance [Ra], [Be2]. The study of the proof reveals that the assumption that $s_0 \in \Sigma \cap\,]a, b[$ is isolated is not necessary. Indeed, denote

$$\Gamma = (\mathbb{R} \setminus \Sigma) \cap [a, b] = \{s \in [a, b] \mid A_s \notin \sigma_M(\mathscr{L})\}$$

and $\deg_{\mathscr{H}}(\mathscr{L} - \mathscr{A}_s, B, 0) = (-1)^{\chi_s}$ for all $s \in \Gamma$. We can write

$$\Gamma = \Gamma_+ \cup \Gamma_-,$$

where $\Gamma_\pm = \{s \in \Gamma \mid (-1)^{\chi_s} = \pm 1\}$. The following general result contains Theorem 6.2 as a special case.

THEOREM 6.3. *Assume that* (6.1) *holds. A given* $s_0 \in \Sigma \cap [a, b]$ *is a point of bifurcation for equation* (E) *if* $s_0 \in \bar{\Gamma}_+ \cap \bar{\Gamma}_-$.

In analogy with above results we can also study bifurcation at infinity. Indeed, we say that $(s; \infty)$ is an *asymptotic point of bifurcation* for equation (E) if there exists a sequence of solutions $((s_j, u_j))_{j=1}^{\infty}$ such that $s_j \to s$ and $\|u\|_j \to \infty$. For instance we get the following variant of Theorem 6.2 (cf. [Be2], [BM5]):

THEOREM 6.4. *Assume that instead of* (6.1) *the nonlinearity satisfies the condition*

$$\|\mathcal{N}(s, u)\| = o(\|u\|) \qquad \text{as } \|u\| \to \infty \text{ uniformly in } s \text{ on } [a, b].$$

If $s \in \Sigma \cap [a, b]$ *is an isolated point and* $\chi_{s-} + \chi_{s+}$ *is odd, then* $(s; \infty)$ *is an asymptotic point of bifurcation for equation* (E).

Note that in space L_2, as for instance in Section 5 of this paper, the condition (6.1) is not relevant for a Nemytskii operator. However, it is reasonable to consider the asymptotic bifurcation for a wave-beam system in an L_2-setting.

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